Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 342 (2008) 255–276

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Multiple solutions for some Schrödinger equations with convex and critical nonlinearities in $\mathbb{R}^N$ ☆

Feizhi Wang

School of Mathematics and Informational Science, Yantai University, Yantai 264005, Shangdong, PR China

Received 20 August 2007

Available online 8 December 2007

Submitted by J. Mawhin

Abstract

In this paper, we consider the existence of multiple solutions for a class of nonlinear Schrödinger equation with indefinite linear part and convex-critical nonlinearities in the whole space $\mathbb{R}^N$.

Keywords: Schrödinger equation; Convex and critical nonlinearities; (∇)-Theorem; Linking theorem; Orthogonalization technique

1. Introduction

In this paper, we consider the following nonlinear Schrödinger problem:

$$-\Delta u + V_\lambda(x)u = a(x)|u|^{p-2}u + K(x)|u|^{2^*-2}u, \quad u \in H^1(\mathbb{R}^N), \tag{$\mathscr{P}_\lambda$}$$

where $N \geqslant 3$, $2 < p < 2^* := 2N/(N-2)$, $V_\lambda(x) = V(x) - \lambda$, $\lambda \in \mathbb{R}$, $V(x) \in C(\mathbb{R}^N, \mathbb{R})$, $a(x)$ and $K(x)$ are bounded positive continuous functions. More precisely, we assume the following hypotheses:

(A) $a(x) \in C(\mathbb{R}^N, \mathbb{R}^+)$ and $\lim_{|x|\to\infty} a(x) = a_\infty > 0$.

(K_1) $K(x) \in C(\mathbb{R}^N, \mathbb{R}^+)$ and $K_m := \inf_{x\in\mathbb{R}^N} K(x) > 0$.

(K_2) $K(x)$ attains its maximum at 0. $K_M := K(0) = \max_{\mathbb{R}^N} K(x)$ and there exists a positive constant α such that $K(x_0) - K(x) = o(|x_0 - x|^\alpha)$.

(V_1) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ and $\lim_{|x|\to\infty} V(x) = v_\infty > 0$.

(V_2) $\sigma(-\Delta + V) \cap (-\infty, 0) \neq \emptyset$ and $0 \notin \sigma(-\Delta + V)$.

☆ This work is supported by the National Natural Science Foundation of China under Grant No. 10671169.
E-mail address: wangfz@ytu.edu.cn.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.12.009

Remark 1.1.

(i) $a(x)$ and $K(x)$ can be positive constants.
(ii) Condition (V_1) means that $\sigma_{ess}(-\Delta+V(x))=[v_\infty,\infty)$ and $\sigma(-\Delta+V(x))\cap(-\infty,v_\infty)=\sigma_p(-\Delta+V(x))$ (see [6]), where $\sigma(-\Delta+V(x))$, $\sigma_{ess}(-\Delta+V(x))$ and $\sigma_p(-\Delta+V(x))$ denote the spectrum, essential spectrum and point spectrum of the Schrödinger operator $-\Delta+V$ acting on $L^2(\mathbb{R}^N)$, respectively. And also, from Theorem 30 in [7, pp. 150], for any $\varepsilon>0$, $\sigma(-\Delta+V(x))$ on $(-\infty,v_\infty-\varepsilon)$ consists of a finite number of eigenvalues with finite multiplicities.
(iii) In the case (V_2), the linear part of problem $(\mathscr{P}_\lambda)$ is indefinite and 0 is in a gap of $\sigma(-\Delta+V(x))$.
(iv) From (V_1)–(V_2), we can conclude that the spectrum of $-\Delta+V(x)$ in this paper has the following character:

(V_{12}) $\lambda_1<\lambda_2\leqslant\cdots\leqslant\lambda_{i-1}<0<\lambda_i=\lambda_{i+1}=\cdots=\lambda_{i+s}<\lambda_{i+s+1}\leqslant\cdots\leqslant\lambda_j\leqslant v_\infty$, where $j\geqslant 2$, $i,s,j\in\mathbb{N}$, and $\lambda_k\in\sigma_p(-\Delta+V(x))$, $k=1,\dots,i+s$.

It is easy to see that: for $\lambda_j<v_\infty$, $\lambda_j\in\sigma_p(-\Delta+V(x))$; for $\lambda_j=v_\infty$, $\lambda_j\in\sigma_{ess}(-\Delta+V(x))$.

By a solution, we understand a function $u\in H^1(\mathbb{R}^N)$ satisfying $(\mathscr{P}_\lambda)$ in weak sense. Obviously, $u=0$ is a trivial solution of $(\mathscr{P}_\lambda)$. We define a functional $I_\lambda: H^1(\mathbb{R}^N)\to\mathbb{R}$ by

$$I_\lambda(u)=\frac{1}{2}\int_{\mathbb{R}^N}\left[|\nabla u|^2+\big(V(x)-\lambda\big)u^2\right]dx-\frac{1}{p}\int_{\mathbb{R}^N}a(x)|u|^p\,dx-\frac{1}{2^*}\int_{\mathbb{R}^N}K(x)|u|^{2^*}\,dx.$$

We can see that $I_\lambda\in C^1(H^1(\mathbb{R}^N),\mathbb{R})$. Therefore solutions of $(\mathscr{P}_\lambda)$ correspond to critical points of the functional I_λ. We denote the strong and the weak convergence in $H^1(\mathbb{R}^N)$ by $\to$ and $\rightharpoonup$, respectively. Set $\|u\|_1:=[\int_{\mathbb{R}^N}(|\nabla u|^2+v_\infty u^2)\,dx]^{1/2}$ and $|u|_q:=\{\int_{\mathbb{R}^N}|u|^q\,dx\}^{1/q}$ for $1<q<\infty$. We say I_λ satisfies the Palais–Smale condition at level c ($(PS)_c$-condition for short), if each (PS)-sequence has a convergent subsequence.

In the following we give our main result:

Theorem 1.1. *Suppose* (V_1)–(V_2), (A), (K_1) *and* (K_2) *hold. Denote by* $\eta:=-V(0)$ *and* $\varsigma=a(0)$, *where* η *and* ς *are positive constants. Then there exists a positive constant* $\delta_0>0$ *such that if* $\lambda\in(\lambda_i-\delta_0,\lambda_i)$ *and* $\lambda_i-\delta_0>0$, *then the problem* $(\mathscr{P}_\lambda)$ *has at least three nontrivial solutions with positive critical values, provided one of the following conditions holds*:

(a) $N=3$, $\alpha\geqslant 1$ *and* $p\in(4,6)$;
(b) $N=3$, $\alpha\geqslant 1$, $p=4$ *and* η *or* ς *large enough*;
(c) $N=3$, $\alpha\in(0,1)$ *and* $p>6-2\alpha$;
(d) $N=3$, $\alpha\in(0,1)$, $p=6-2\alpha$ *and* ς *large enough*;
(e) $N\geqslant 4$ *and* $\alpha\geqslant 2$;
(f) $N\geqslant 4$, $\alpha\in(0,2)$ *and* $p>\frac{2(N-\alpha)}{N-2}$;
(g) $N\geqslant 4$, $\alpha\in(0,2)$, $p=\frac{2(N-\alpha)}{N-2}$ *and* ς *large enough.*

Remark 1.2. Through the linking theorem due to Rabinowitz (see [18, Theorem 2.12]), under the following conditions:

(i) (V_1) holds;
(ii) $\lambda\in(-\infty,v_\infty)$ and $\lambda\notin\sigma_p(-\Delta+V)$;
(iii) One of the following cases:
 (a) $N=3$, $\alpha\geqslant 1$ and $p\in(4,6)$,
 (b) $N=3$, $\alpha\geqslant 1$, $p=4$ and η or ς large enough,
 (c) $N=3$, $\alpha\in(0,1)$ and $p>6-2\alpha$,
 (d) $N=3$, $\alpha\in(0,1)$, $p=6-2\alpha$ and ς large enough,
 (e) $N\geqslant 4$ and $\alpha\geqslant 2$,
 (f) $N\geqslant 4$, $\alpha\in(0,2)$ and $p>\frac{2(N-\alpha)}{N-2}$,
 (g) $N\geqslant 4$, $\alpha\in(0,2)$, $p=\frac{2(N-\alpha)}{N-2}$ and ς large enough,

one nontrivial solution of problem $(\mathscr{P}_\lambda)$ can be obtained. In this case $V(x)$ may be a positive constant, i.e., $-\infty < \lambda < V(x) \equiv v_\infty$.

From now on, we always set $\lambda \in (0, \lambda_i)$. For simplicity we assume that $v_\infty = 1$, $a_\infty = 1$ and $K_M = 1$. Then by (V_{12}), $0 < \lambda < \lambda_i < v_\infty = 1$.

Recently, in [11], Li–Wang–Zeng considered the following problem:

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{P1}$$

where $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^N} V(x) > 0$. They thought about two cases of the potential: $V(x)$, one is periodic, i.e., the x-dependence is periodic; the other is when $V(x)$ has a bounded potential well in the sense that $\lim_{|x|\to\infty} V(x)$ exists and is equal to $\sup_{x\in\mathbb{R}^N} V(x)$. In these cases the linear part of problem (P1) is positive definite, i.e., $\sigma(-\Delta + V(x)) > 0$. Furthermore, under the following super-quadratic condition:

(SQ) $\lim_{|u|\to\infty} \frac{F(x,u)}{u^2} = \infty$, uniformly in x, where $F(x,u) = \int_0^u f(x,t)\,dt$,

and a standard Nehari type condition they gave the existence of ground state solutions for problem (P1). But if the linear part is indefinite, the method which appear in [11] is invalid.

Indeed, there is a large number of works have been devoted to the problem like $(\mathscr{P}_\lambda)$ with the indefinite linear part. We refer the readers to [1–6,16,17] and references therein. Especially, in [16], Wang considered the problem $(\mathscr{P}_\lambda)$ without the critical term, i.e., $K(x) \equiv 0$. He obtained the existence of multiple solutions through a Linking theorem due to Rabinowitz (see [18, Theorem 2.12]), and a (∇)-theorem which was initiated in [13] and then developed and applied in many situations to obtain multiplicity results, for example see [12,14], and references therein. In the present paper, under condition (K_1) we continue to use the method of [16] to study the problem $(\mathscr{P}_\lambda)$. Because the problem contains convex and critical nonlinearities in $\mathbb{R}^N$, there are more difficulties to overcome.

In the present paper Lemmas 3.2 and 6.1 are crucial in our approach. In Lemma 3.2 we proof the functional I_λ satisfies the PS condition in a certain compactness range. This range is $(0, \mathfrak{S})$, where

$$\mathfrak{S} := \min_{t\in[0,1]} f(t) = \min_{t\in[0,1]} \left\{ \left(\frac{1}{2} - \frac{1}{p}\right)(t\mathcal{S}_{\lambda,p})^{p/(p-2)} + \frac{1}{N}\big[(1-t)\mathbb{S}\big]^{N/2} \right\}.$$

In the above formula, $\mathbb{S}$ and $\mathcal{S}_{\lambda,p}$ are well-known Sobolev constants

$$\mathbb{S} := \inf\left\{ \frac{|\nabla u|_2^2}{|u|_{2^*}^2} \colon u \in H^1(\mathbb{R}^N) \setminus \{0\} \right\}$$

and

$$\mathcal{S}_{\lambda,p} := \inf\left\{ \frac{\int_{\mathbb{R}^N} [|\nabla u|^2 + (1-\lambda)u^2]\,dx}{|u|_p^2} \colon u \in H^1(\mathbb{R}^N) \setminus \{0\} \right\}.$$

In Lemma 6.1, for certain combined Sobolev approximating functions, which belong to $H^1(\mathbb{R}^N)$, we show that its value of I_λ stays in $(0, \mathfrak{S})$. We call combined Sobolev approximating functions the sum of two truncations of positive radial entire functions which achieve the best constants: $\mathbb{S}$ and $\mathcal{S}_{\lambda,p}$, respectively.

This paper is organized as follows: In Section 2, we make some preliminaries. In Section 3, we show functional I_λ satisfies the $(\mathrm{PS})_c$-condition for some c. In Sections 4 and 5, we prove that I_λ has two linking geometrical structure. In Section 6, we give the proof of Theorem 1.1. In Appendix A, two linking theorems are given. In Appendix B, we give the proof of lemmas of Section 4. For the readers' convenience and the completeness of paper, some parts similar to [16] will appear again in our paper.

2. Preliminaries

We denote by ψ_k the eigenfunction in $H^1(\mathbb{R}^N)$ corresponding to the eigenvalue λ_k, $k = 1, \dots, i+s$. Set

$$X_1 = \mathrm{span}\{\psi_k,\ k = 1, 2, \dots, i-1\},$$
$$X_2 = \mathrm{span}\{\psi_k,\ k = i, \dots, i+s\},$$
$$X_3 = (X_1 \oplus X_2)^{\perp}. \tag{2.1}$$

Then $X_1 \oplus X_2 \oplus X_3 = H^1(\mathbb{R}^N)$. Denote by P_l and $P_{l_1 l_2}$ the orthogonal projections of $H^1(\mathbb{R}^N)$ onto X_l and $X_{l_1} \oplus X_{l_2}$, respectively, where $l, l_1, l_2 \in \{1, 2, 3\}$ and $l_1 \leqslant l_2$. The quadratic form $\int_{\mathbb{R}^{\mathbb{N}}}[|\nabla u|^2 + V(x)u^2]\,dx$ is negative definite on X_1 and positive definite on $X_2 \oplus X_3$. Using arguments similar to those in the proof of Lemma 1.2 in [6] and noting that $0 < \lambda < \lambda_i < v_\infty = 1$, we can define two new norms $\|\cdot\|$ and $\|\cdot\|_\lambda$ on X_1 and $X_2 \oplus X_3$ by setting

$$\|P_{23}u\|^2 - \|P_1 u\|^2 = \int_{\mathbb{R}^{\mathbb{N}}} \big[|\nabla u|^2 + V(x)u^2\big]\,dx, \quad u \in H^1\big(\mathbb{R}^N\big),$$

and

$$\|P_{23}u\|_\lambda^2 - \|P_1 u\|_\lambda^2 = \int_{\mathbb{R}^{\mathbb{N}}} \big[|\nabla u|^2 + \big(V(x) - \lambda\big)u^2\big]\,dx, \quad u \in H^1\big(\mathbb{R}^N\big).$$

$\|\cdot\|$ and $\|\cdot\|_\lambda$ are equivalent to $\|\cdot\|_0$ since $0 \notin \sigma(-\Delta + V(x))$ and $0 \notin \sigma(-\Delta + V(x) - \lambda)$, respectively.

It is well known that $\mathbb{S}$ is attained by the functions

$$u_\varepsilon(x) = \big[N(N-2)\big]^{(N-2)/4}\left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{(N-2)/2}, \quad \forall x \in \mathbb{R}^N,\ \varepsilon > 0. \tag{2.2}$$

From the well-known results of [2,9,10,15], it follows that the following problem:

$$-\Delta u + (1-\lambda)u = u^{p-1}, \quad u > 0 \quad \text{in } \mathbb{R}^N,$$

has a unique positive solution $w(x) \in H^1(\mathbb{R}^N)$ up to a translation, and such that

$$\int_{\mathbb{R}^{\mathbb{N}}} \big[\big|\nabla w(x)\big|^2 + (1-\lambda)w^2(x)\big]\,dx = \mathcal{S}_{\lambda,p}, \qquad \big|w(x)\big|_p^p = 1,$$
$$\lim_{|x|\to\infty} w(x)|x|^{(N-1)/2}\exp\big\{(1-\lambda)^{-1/2}|x|\big\} = d_1 > 0 \tag{2.3}$$

and

$$\lim_{|x|\to\infty} \big|\nabla w(x)\big||x|^{(N-1)/2}\exp\big\{(1-\lambda)^{-1/2}|x|\big\} = d_2 > 0,$$

for some positive constants d_1 and d_2.

Remark 2.1. From the definition of $\mathcal{S}_{\lambda,p}$, it is easy to see that $\mathcal{S}_{\lambda,p} \geqslant \mathcal{S}_{\lambda_i,p} > 0$, for $\lambda \in (0, \lambda_i)$. $\mathcal{S}_{\lambda_i,p}$ is independent on λ.

About $\mathbb{S}$ and $\mathcal{S}_{0,p}$, we give a lemma:

Lemma 2.1. *$\mathbb{S}^\mu \geqslant \mathcal{S}_{0,p}$, where $\mu := \frac{p-2}{2^*-2}\frac{2^*}{p} = \frac{N}{2}\frac{p-2}{p}$.*

Proof. Hölder and Sobolev inequalities imply that

$$|u|_p \leqslant |u|_2^{1-\mu} \cdot |u|_{2^*}^{\mu} \leqslant |u|_2^{1-\mu}\mathbb{S}^{-\mu/2}|\nabla u|_2^{\mu} \leqslant \|u\|_0^{1-\mu}\mathbb{S}^{-\mu/2}\|u\|_0^{\mu} = \mathbb{S}^{-\mu/2}\|u\|_0.$$

So $|u|_p^2 \leqslant \mathbb{S}^{-\mu}\|u\|_0^2$. By the definition of $\mathcal{S}_{0,p}$, we have $\mathbb{S}^\mu \geqslant \mathcal{S}_{0,p}$. □

3. The $(\mathrm{PS})_c$-condition

Lemma 3.1. *Suppose $\{u_n\}$ is a $(\mathrm{PS})_c$-sequence for I_λ. If $c>0$, then $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$.*

Proof. By contradiction, let us assume that $\|u_n\|_\lambda \to \infty$. Define $w_n = u_n/\|u_n\|_\lambda$, so $\|w_n\|_\lambda = 1$. Then there is w in $H^1(\mathbb{R}^N)$ such that $w_n \rightharpoonup w$ (passing to a subsequence, if necessary).

Case (i): $w \neq 0$. By (V_1), there exists a constant $M>0$ such that for all n,

$$\int_{\mathbb{R}^N} \left[|\nabla w_n|^2 + \big(V(x)-\lambda\big)w_n^2\right] dx \leqslant M. \tag{3.1}$$

From $I(u_n) \to c$, we have

$$\frac{1}{2}\int_{\mathbb{R}^N} \left[|\nabla w_n|^2 + \big(V(x)-\lambda\big)w_n^2\right] dx - \int_{\mathbb{R}^N} \frac{\frac{a(x)}{p}|u_n|^p + \frac{K(x)}{2^*}|u_n|^{2^*}}{\|u_n\|_\lambda^2}\, dx = \frac{c+o(1)}{\|u_n\|_\lambda^2} \geqslant 0.$$

By (3.1), we have (passing to a subsequence, if necessary)

$$\frac{M}{2} \geqslant \lim_{n\to\infty} \frac{1}{p}\int_{\mathbb{R}^N} \frac{a(x)|u_n|^p}{\|u_n\|_\lambda^2}\, dx = \frac{1}{p}\lim_{n\to\infty}\left(\int_{\mathbb{R}^N} a(x)|w_n|^p\, dx \cdot \|u_n\|_\lambda^{p-2}\right).$$

Since $\|u_n\|_\lambda \to \infty$ and $p>2$, we have $\lim_{n\to\infty}\int_{\mathbb{R}^N} a(x)|w_n|^p\, dx = 0$. By $a(x)$ is positive bounded and $w_n \rightharpoonup w$, $w=0$. A contradiction arises.

Case (ii): $w=0$. We have

$$\frac{I_\lambda(u_n)}{\|u_n\|_\lambda^2} = \frac{1}{2}\int_{\mathbb{R}^N} \left[|\nabla w_n|^2 + \big(V(x)-\lambda\big)w_n^2\right] dx - \frac{1}{\|u_n\|_\lambda^2}\int_{\mathbb{R}^N}\left[\frac{a(x)}{p}|u_n|^p + \frac{K(x)}{2^*}|u_n|^{2^*}\right] dx$$

and

$$\left\langle \frac{I_\lambda'(u_n)}{\|u_n\|_\lambda}, w_n\right\rangle = \int_{\mathbb{R}^N} |\nabla w_n|^2\, dx + \int_{\mathbb{R}^N} \big(V(x)-\lambda\big)w_n^2\, dx - \frac{\int_{\mathbb{R}^N}[a(x)|u_n|^p + K(x)|u_n|^{2^*}]\, dx}{\|u_n\|_\lambda^2}.$$

Since $w_n \rightharpoonup 0$ and $\lim_{|x|\to\infty} V(x) = 1$, we obtain

$$\int_{\mathbb{R}^N} V(x)w_n^2\, dx = \int_{\mathbb{R}^N} w_n^2\, dx + o(1).$$

Therefore, for n large enough,

$$\lim_{n\to\infty} \frac{1}{\|u_n\|_\lambda^2}\int_{\mathbb{R}^N}\left[\frac{a(x)}{p}|u_n|^p + \frac{K(x)}{2^*}|u_n|^{2^*}\right] dx = \frac{1}{2}$$

and

$$\lim_{n\to\infty} \frac{1}{\|u_n\|_\lambda^2}\int_{\mathbb{R}^N}\left[a(x)|u_n|^p + K(x)|u_n|^{2^*}\right] dx = 1.$$

Then

$$\begin{aligned}
\frac{1}{p} &= \lim_{n\to\infty} \frac{1}{\|u_n\|_\lambda^2}\int_{\mathbb{R}^N}\left[\frac{a(x)}{p}|u_n|^p + \frac{K(x)}{p}|u_n|^{2^*}\right] dx \\
&\geqslant \lim_{n\to\infty} \frac{1}{\|u_n\|_\lambda^2}\int_{\mathbb{R}^N}\left[\frac{a(x)}{p}|u_n|^p + \frac{K(x)}{2^*}|u_n|^{2^*}\right] dx \\
&= \frac{1}{2}.
\end{aligned}$$

Thus $p \leqslant 2$. This is a contradiction also.
So $\{u_n\}$ is bounded. $\square$

Lemma 3.2. *Suppose the assumptions of Theorem* 1.1 *hold. Let* $\{u_n\} \subset H^1(\mathbb{R}^N)$ *be a* $(\mathrm{PS})_c$*-sequence for* I_λ *with*

$$0 < c < \mathfrak{S} := \min_{t\in[0,1]} f(t), \tag{3.2}$$

where

$$f(t) := \left(\frac{1}{2} - \frac{1}{p}\right)(t\mathcal{S}_{\lambda,p})^{p/(p-2)} + \frac{1}{N}\big[(1-t)\mathbb{S}\big]^{N/2} \quad \text{and} \quad t \in [0,1],$$

then there exists a subsequence of $\{u_n\}$ *which converges strongly in* $H^1(\mathbb{R}^N)$.

Proof. It follows from Lemma 3.1 that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Going if necessary to a subsequence, we may assume $u_n \rightharpoonup v$. Then we have $\langle I'_\lambda(v), v\rangle = 0$, i.e.,

$$\int_{\mathbb{R}^N} \big[|\nabla v|^2 + \big(V(x)-\lambda\big)v^2\big]\,dx = \int_{\mathbb{R}^N} a(x)|v|^p\,dx + K(x)\int_{\mathbb{R}^N} |v|^{2^*}\,dx.$$

So

$$\begin{aligned} I_\lambda(v) &= \frac{1}{2}\int_{\mathbb{R}^N} \big[|\nabla v|^2 + \big(V(x)-\lambda\big)v^2\big]\,dx - \frac{1}{p}\int_{\mathbb{R}^N} a(x)|v|^p\,dx - \frac{1}{2^*}\int_{\mathbb{R}^N} K(x)|v|^{2^*}\,dx \\ &= \left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^N} a(x)|v|^p\,dx + \frac{1}{N}\int_{\mathbb{R}^N} K(x)|v|^{2^*}\,dx \\ &\geqslant 0. \end{aligned} \tag{3.3}$$

Let $v_n = u_n - v$. By Lemma 1.32 of [18], $\lim_{|x|\to\infty} V(x) = v_\infty\ (=1)$ and $\lim_{|x|\to\infty} a(x) = a_\infty\ (=1)$,

$$\begin{aligned} &\int_{\mathbb{R}^N} \big[|\nabla u_n|^2 + \big(V(x)-\lambda\big)u_n^2\big]\,dx \\ &\quad= \int_{\mathbb{R}^N} \big[|\nabla v_n|^2 + \big(V(x)-\lambda\big)v_n^2\big]\,dx + \int_{\mathbb{R}^N} \big[|\nabla v|^2 + \big(V(x)-\lambda\big)v^2\big]\,dx + o(1) \\ &\quad= \int_{\mathbb{R}^N} \big[|\nabla v_n|^2 + (1-\lambda)v_n^2\big]\,dx + \int_{\mathbb{R}^N} \big[|\nabla v|^2 + \big(V(x)-\lambda\big)v^2\big]\,dx + o(1), \end{aligned} \tag{3.4}$$

$$\int_{\mathbb{R}^N} a(x)|u_n|^p\,dx = \int_{\mathbb{R}^N} |v_n|^p\,dx + \int_{\mathbb{R}^N} a(x)|v|^p\,dx + o(1) \tag{3.5}$$

and

$$\int_{\mathbb{R}^N} K(x)|u_n|^{2^*}\,dx = \int_{\mathbb{R}^N} K(x)|v_n|^{2^*}\,dx + \int_{\mathbb{R}^N} K(x)|v|^{2^*}\,dx + o(1). \tag{3.6}$$

Then

$$\begin{aligned} \big\langle I'_\lambda(u_n), u_n\big\rangle &= \int_{\mathbb{R}^N} \big[|\nabla u_n|^2 + \big(V(x)-\lambda\big)u_n^2\big]\,dx - \int_{\mathbb{R}^N} a(x)|u_n|^p\,dx - \int_{\mathbb{R}^N} K(x)|u_n|^{2^*}\,dx \\ &= \int_{\mathbb{R}^N} \big[|\nabla v_n|^2 + (1-\lambda)v_n^2\big]\,dx + \int_{\mathbb{R}^N} \big[|\nabla v|^2 + \big(V(x)-\lambda\big)v^2\big]\,dx \end{aligned}$$

$$-\int_{\mathbb{R}^N} |v_n|^p\,dx - \int_{\mathbb{R}^N} |v|^p\,dx - \int_{\mathbb{R}^N} K(x)|v_n|^{2^*}\,dx - \int_{\mathbb{R}^N} K(x)|v|^{2^*}\,dx + o(1).$$

By $\langle I'_\lambda(v), v\rangle = 0$ and $\langle I'_\lambda(u_n), u_n\rangle \to 0$, we may therefore assume that

$$\int_{\mathbb{R}^N} \big[|\nabla v_n|^2 + (1-\lambda)v_n^2\big]\,dx \to b$$

and

$$\int_{\mathbb{R}^N} |v_n|^p\,dx + \int_{\mathbb{R}^N} K(x)|v_n|^{2^*}\,dx \to b. \tag{3.7}$$

We claim that $b \neq 0$ is impossible. In the following we assume that $b \neq 0$. By (3.7), we can assume that: either $\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)|v_n|^{2^*}\,dx = 0$ or $\int_{\mathbb{R}^N} K(x)|v_n|^{2^*}\,dx \neq 0$.

Case (i): $\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)|v_n|^{2^*}\,dx = 0$. By the following inequality:

$$|u|_p \leqslant |u|_2^{1-\mu}\cdot|u|_{2^*}^{\mu},$$

where $\mu := \frac{N}{2}\frac{p-2}{p}$, we obtain $\lim_{n\to\infty}\int_{\mathbb{R}^N} |v_n|^p\,dx = 0$. This is a contradiction with $b \neq 0$.

Case (ii): $\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)|v_n|^{2^*}\,dx \neq 0$. We can assume that: either $\lim_{n\to\infty}\int_{\mathbb{R}^N} |v_n|^p\,dx = 0$ or $\lim_{n\to\infty}\int_{\mathbb{R}^N} |v_n|^p\,dx \neq 0$.

(a) $\lim_{n\to\infty}\int_{\mathbb{R}^N} |v_n|^p\,dx = 0$. By the definition of $\mathbb{S}$,

$$\int_{\mathbb{R}^N} \big[|\nabla v_n|^2 + (1-\lambda)v_n^2\big]\,dx \geqslant \frac{\mathbb{S}}{K_M^{(N-2)/N}}\bigg(\int_{\mathbb{R}^N} K(x)|v_n|^{2^*}\,dx\bigg)^{2/2^*},$$

which yields $b \geqslant \mathbb{S}K_M^{(2-N)/N} b^{(N-2)/N}$. Noting that $K_M = \sup_{x\in\mathbb{R}^N} K(x) = 1$, $b \geqslant \mathbb{S}^{N/2}$. By (3.3)–(3.6),

$$\begin{aligned}
I_\lambda(u_n) &= \frac{1}{2}\int_{\mathbb{R}^N} \big[|\nabla u_n|^2 + \big(V(x)-\lambda\big)u_n^2\big]\,dx - \frac{1}{p}\int_{\mathbb{R}^N} a(x)|u_n|^p\,dx - \frac{1}{2^*}\int_{\mathbb{R}^N} K(x)|u_n|^{2^*}\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}^N} \big[|\nabla v_n|^2 + (1-\lambda)v_n^2\big]\,dx + \frac{1}{2}\int_{\mathbb{R}^N} \big[|\nabla v|^2 + \big(V(x)-\lambda\big)v^2\big]\,dx \\
&\quad - \frac{1}{p}\int_{\mathbb{R}^N} |v_n|^p\,dx - \frac{1}{p}\int_{\mathbb{R}^N} a(x)|v|^p\,dx - \frac{1}{2^*}\int_{\mathbb{R}^N} K(x)|v_n|^{2^*}\,dx - \frac{1}{2^*}\int_{\mathbb{R}^N} K(x)|v|^{2^*}\,dx + o(1) \\
&= I(v) + \frac{1}{2}\int_{\mathbb{R}^N} \big[|\nabla v_n|^2 + (1-\lambda)v_n^2\big]\,dx - \frac{1}{p}\int_{\mathbb{R}^N} |v_n|^p\,dx - \frac{1}{2^*}\int_{\mathbb{R}^N} K(x)|v_n|^{2^*}\,dx + o(1).
\end{aligned}$$

It follows from $b \geqslant \mathbb{S}^{N/2}$ that

$$c \geqslant \frac{1}{N}b \geqslant \frac{1}{N}\mathbb{S}^{N/2}.$$

(b) $\lim_{n\to\infty}\int_{\mathbb{R}^N} |v_n|^p\,dx \neq 0$. We assume that (passing to a subsequence, if necessary) $\lim_{n\to\infty}\int_{\mathbb{R}^N} |v_n|^p\,dx = tb$ and $\lim_{n\to\infty}\int_{\mathbb{R}^N} |v_n|^{2^*}\,dx = (1-t)b$, where $t \in (0,1)$ is a constant. Then by the definition of $\mathcal{S}_{\lambda,p}$ and $\mathbb{S}$, we have

$$b \geqslant \mathcal{S}_{\lambda,p}(tb)^{2/p} \quad \text{and} \quad b \geqslant \mathbb{S}\big[(1-t)b\big]^{2/2^*}.$$

Thus $tb \geqslant (t\mathcal{S}_{\lambda,p})^{p/(p-2)}$ and $(1-t)b \geqslant [(1-t)\mathbb{S}]^{N/2}$. Similarly to Case (i), we have

$$c \geqslant \bigg(\frac{1}{2}-\frac{1}{p}\bigg)(t\mathcal{S}_{\lambda,p})^{p/(p-2)} + \frac{1}{N}\big[(1-t)\mathbb{S}\big]^{N/2}.$$

(a) and (b) imply that

$$c \geqslant \mathfrak{S} := \min_{t\in[0,1]}\left\{\left(\frac{1}{2}-\frac{1}{p}\right)(t\mathcal{S}_{\lambda,p})^{p/(p-2)} + \frac{1}{N}\big[(1-t)\mathbb{S}\big]^{N/2}\right\},$$

which contradict with (3.2).

Combining Case (i) and Case (ii), we get $b = 0$.

Thus

$$0 \leqslant \|v_n\|_\lambda = \left\{\int_{\mathbb{R}^N}\big[|\nabla v_n|^2 + (1-\lambda)v_n^2\big]\,dx\right\}^{1/2} \to b^{1/2} = 0,$$

which gives $v_n \to 0$ in $H^1(\mathbb{R}^N)$. We complete the proof. □

Remark 3.1. Since

$$\mathfrak{S} := \min_{t\in[0,1]} f(t) := \min_{t\in[0,1]}\left\{\left(\frac{1}{2}-\frac{1}{p}\right)(t\mathcal{S}_{\lambda,p})^{p/(p-2)} + \frac{1}{N}\big((1-t)\mathbb{S}\big)^{N/2}\right\},$$

there exists a number $\tau_0 \in (0,1)$ such that $\mathfrak{S} := f(\tau_0)$.

4. Linking structure I

This section is very similar to [16, Section 2]. We only give an outline. The proof of lemmas can be founded in Appendix B.

Set

$$S_{23}(\rho) = \big\{u \in X_2 \oplus X_3\colon\ \|u\| = \rho\big\}$$

and

$$T_{1,2}(R) = \big\{u \in X_1 \oplus X_2\colon\ \|u\| = R\big\} \cup \big\{u \in X_1\colon\ \|u\| \leqslant R\big\},$$

where X_1, X_2 and X_3 are given in (2.1).

Then we have

Lemma 4.1. *Assume $0 < \lambda < \lambda_i$. Then there exist R and ρ such that*

$$\sup I_\lambda\big(T_{1,2}(R)\big) < \inf I_\lambda\big(S_{23}(\rho)\big),$$

where $0 < \rho < R$.

Lemma 4.2. *Assume* (V_1) *and* (V_2). *Then $\exists \varepsilon_1 > 0$ such that $\forall\lambda \in (0,\lambda_i)$, the only critical point u of I_λ constrained on $X_1 \oplus X_3$ such that $I_\lambda(u) \in [-\varepsilon_1, \varepsilon_1]$ is the trivial one.*

Lemma 4.3. *Suppose* (V_1), (V_2), *$\lambda \in (0,\lambda_i)$ and $\{u_n\}$ in $H^1(\mathbb{R}^N)$ is such that $I_\lambda(u_n)$ is bounded, $P_2 u_n \to 0$ and $P_{13} I'_\lambda(u_n) \to 0$. Then $\{u_n\}$ is bounded.*

Lemma 4.4. *Assume* (V_1), (V_2) *and $\lambda \in (0,\lambda_i)$. Then $\forall\delta > 0$, $\exists\varepsilon_0 > 0$ such that $\forall\lambda \in (\lambda_i - \delta, \lambda_i)$ and $\forall\varepsilon', \varepsilon'' \in (0,\varepsilon_0)$, $\varepsilon' < \varepsilon''$, the condition $(\nabla)(I_\lambda, X_1 \oplus X_3, \varepsilon', \varepsilon'')$ holds.*

5. Linking structure II

In this section, we denote $B_r(0)$ by B_r, where $B_r(0) = \{x \in \mathbb{R}^N\colon\ |x| \leqslant r\}$. Let $X = X_1 \oplus X_2$ and $W = X_3$, where X_1, X_2 and X_3 are given in (2.1). Note $X \oplus W = H^1(\mathbb{R}^N)$.

For each $m \in \mathbb{N}$ ($m \geqslant 3$), we define a function $\xi_m : \mathbb{R}^N \to \mathbb{R}$ as

$$\xi_m(x) := \begin{cases} 0 & \text{if } x \in B_{1/m}, \\ m|x| - 1 & \text{if } x \in A_m = B_{2/m} \setminus B_{1/m}, \\ 1 & \text{if } x \in B_m \setminus B_{2/m}, \\ m + 1 - |x| & \text{if } x \in D_m = B_{m+1} \setminus B_m, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus B_{m+1}. \end{cases}$$

Set $\psi_i^m := \xi_m \psi_i$ and

$$X_m = \operatorname{span}\{\psi_k^m,\ k = 1, 2, \ldots, i + s\},$$

where ψ_k is the eigenfunction of λ_k. Then similar to Lemma 3.1 in [17], we can obtain that

(i) For $k \in \{1, 2, \ldots, i + s\}$,

$$\psi_k^m \to \psi_k \quad \text{in } H^1(\mathbb{R}^N) \text{ as } m \to \infty.$$

(ii) There exist $M_1, \varrho > 0$ such that for all $m > M_1$,

$$\max_{\{u \in X_m \mid \int_{\mathbb{R}^{\mathbb{N}}} u^2 = 1\}} \int_{\mathbb{R}^{\mathbb{N}}} |\nabla u|^2 + V(x)u^2\,dx \leqslant \lambda_{i+s} + \varrho < \lambda_{i+s+1} \leqslant \lambda_j.$$

Furthermore, as Lemma 3.2 in [17] there exists a positive integer M_2 such that for each $m > M_2$,

$$P_{12}(X_m) = X, \qquad \dim(X_m) = i + s \quad \text{and} \quad H^1(\mathbb{R}^N) = X_m \oplus W.$$

In the rest of our paper, we suppose that $m > M_2$.

For each $m \in \mathbb{N}$ ($m \geqslant 3$), we define two functions $\mu_m : \mathbb{R}^N \to \mathbb{R}$ and $\eta_m : \mathbb{R}^N \to \mathbb{R}$ by

$$\mu_m(x) := \begin{cases} 1 & \text{if } x \in B_{1/2m}, \\ 2 - 2m|x| & \text{if } x \in A_m = B_{1/m} \setminus B_{1/2m}, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus B_{1/m} \end{cases}$$

and

$$\eta_m(x) := \begin{cases} 0 & \text{if } x \in B_{m+1}, \\ |x| - m - 1 & \text{if } x \in D_m = B_{m+2} \setminus B_{m+1}, \\ 1 & \text{if } x \in \mathbb{R}^N \setminus B_{m+2}, \end{cases}$$

respectively. Then set

$$U_{m,\varepsilon}(x) := \mu_m(x) \cdot u_\varepsilon(x)$$

and

$$W_{m,z_m}(x) := \eta_m(x) \cdot w_{z_m}(x) = \eta_m(x) \cdot w(x - z_m),$$

where $z_m \in (\mathbb{R}^N \setminus B_{2m}) := \{x \in \mathbb{R}^N \colon |x| \geqslant 2m\}$, $u_\varepsilon(x)$ and $w(x)$ appear in (2.2) and (2.3), respectively.

By the definitions of $U_{m,\varepsilon}(x)$, $W_{m,z_m}(x)$ and X_m, we have

$$\operatorname{supp}(U_{m,\varepsilon}(x)) \cap \operatorname{supp}(v) = \emptyset \quad \text{and} \quad \operatorname{supp}(W_{m,z_m}(x)) \cap \operatorname{supp}(v) = \emptyset,$$

where $v \in X_m$ and $\varepsilon > 0$.

It is well known that the following asymptotic estimates hold as ε small enough:

$$|\nabla U_{m,\varepsilon}|_2^2 = \mathbb{S}^{N/2} + O(\varepsilon^{N-2}),$$
$$|U_{m,\varepsilon}|_{2^*}^{2^*} = \mathbb{S}^{N/2} + O(\varepsilon^N) \tag{5.1}$$

and

$$|U_{m,\varepsilon}|_2^2 = \begin{cases} b\varepsilon^2 + O(\varepsilon^{N-2}) & \text{if } N \geqslant 5,\\ b\varepsilon^2|\log\varepsilon| + O(\varepsilon^2) & \text{if } N = 4,\\ b\varepsilon + O(\varepsilon^2) & \text{if } N = 3,\end{cases} \tag{5.2}$$

where b is a positive constant.

It follows from (K_1) that there exists a positive integer M_3 such that for $m > M_3$,

$$K_M - K(x) \leqslant c|x|^\alpha \quad \text{a.e. } x \in B_{1/m} = B_{1/m}(0).$$

Lemma 5.1. *If $m \geqslant M_3$ and $K(x)$ satisfies* (K_2)*, then for ε small enough*

$$\int_{\mathbb{R}^{\mathbb{N}}} K(x)|u_\varepsilon|^{2^*}\,dx = K_M \mathbb{S}^{N/2} + O\big(h(\varepsilon)\big),$$

where

$$h(\varepsilon) = \begin{cases} \varepsilon^\alpha & \textit{for } \alpha < N,\\ \varepsilon^N|\ln\varepsilon| & \textit{for } \alpha = N,\\ \varepsilon^N & \textit{for } \alpha > N.\end{cases}$$

Proof. See Lemma 2 in [8]. □

Since $V(0) = -\eta < 0$ and $a(0) = \varsigma > 0$, there exists a positive integer M_4 such that $V(x) < -\eta/2 < 0$ and $a(x) > \varsigma/2$ for all $x \in B_{2/m}$ and $m > M_4$. In the rest of our paper, we suppose that $m \geqslant M_0 := \max\{M_1, M_2, M_3, M_4\}$.

Lemma 5.2. *Suppose the assumptions of Theorem* 1.1 *hold. If $\lambda \in (0, \lambda_i)$, then*

$$\limsup_{\lambda\to\lambda_i^-} I_\lambda(X_m) = 0$$

and

$$\limsup_{\lambda\to\lambda_i^-} I_\lambda(X) = 0.$$

Proof. See Lemma 2.2 in [16]. □

Set

$$Z_{m,\varepsilon,z_m} = \tau_0 W_{m,z_m}(x) + (1-\tau_0)U_{m,\varepsilon}(x),$$

where τ_0 appears in Remark 3.1. Let

$$Q_m^{\varepsilon,z_m,R} = \big\{sy + tZ_{m,\varepsilon,z_m}\colon\ y \in X_m,\ \|y\| = 1,\ 0 \leqslant s,t \leqslant R\big\}.$$

Lemma 5.3. *Suppose the assumptions of Theorem* 1.1 *hold. There exist $\delta_1 > 0, \varepsilon_1 > 0$ and $R_1 > \rho$ such that for all $\lambda \in (\lambda_i - \delta_1, \lambda_i), m \geqslant M_0, \varepsilon \in (0, \varepsilon_1)$ and $z_m \geqslant 2m$,*

$$\inf I_\lambda\big(S_3(\rho)\big) > \sup I_\lambda\big(\partial\big(Q_m^{\varepsilon,z_m,R_1}\big)\big)$$

and

$$\inf I_\lambda\big(S_3(\rho)\big) > c_{14}\rho^2, \tag{5.3}$$

where $S_3(\rho) = \{v \in W\colon\ \|v\| = \rho\}$, ρ and $c_{14} > 0$ are independent of λ.

Proof. See Lemma 3.4 in [17] and Lemma 2.3 in [16]. □

6. The proof of Theorem 1.1

Lemma 6.1. *There exists a positive constant $\bar{m} \geqslant M_0$ such that for each $m > \bar{m}$ the following holds: there exist positive constants $R_m \geqslant 2m$ and $\varepsilon_0 > 0$ such that if $|z_m| \geqslant R_m$ and $\varepsilon \in (0, \varepsilon_0)$, then*

$$I_\lambda(tZ_{m,\varepsilon,z_m}) < \mathfrak{S} := \left(\frac{1}{2} - \frac{1}{p}\right)(\tau_0 \mathcal{S}_{\lambda,p})^{p/(p-2)} + \frac{1}{N}\big[(1-\tau_0)\mathbb{S}\big]^{N/2}, \quad \text{for all } t \geqslant 0, \tag{6.1}$$

provided one of the following conditions holds:

(a) $N = 3$, $\alpha \geqslant 1$ *and* $p \in (4, 6)$;
(b) $N = 3$, $\alpha \geqslant 1$, $p = 4$ *and* η *or* ς *large enough*;
(c) $N = 3$, $\alpha \in (0, 1)$ *and* $p > 6 - 2\alpha$;
(d) $N = 3$, $\alpha \in (0, 1)$, $p = 6 - 2\alpha$ *and* ς *large enough*;
(e) $N \geqslant 4$ *and* $\alpha \geqslant 2$;
(f) $N \geqslant 4$, $\alpha \in (0, 2)$ *and* $p > \frac{2(N-\alpha)}{N-2}$;
(g) $N \geqslant 4$, $\alpha \in (0, 2)$, $p = \frac{2(N-\alpha)}{N-2}$ *and* ς *large enough.*

Proof. By the definitions of Z_{m,ε,z_m}, W_{m,z_m} and $U_{m,\varepsilon}$, we have

$$\begin{aligned} I_\lambda(tZ_{m,\varepsilon,z_m}) &= I_\lambda\big(t\big[\tau_0 W_{m,z_m} + (1-\tau_0)U_{m,\varepsilon}\big]\big) \\ &= I_\lambda(t\tau_0 W_{m,z_m}) + I_\lambda\big(t(1-\tau_0)U_{m,\varepsilon}\big). \end{aligned}$$

In order to prove (6.1) it is sufficient to show that there exists a positive constant $\bar{m} \geqslant M_0$ such that for each $m > \bar{m}$ and $t > 0$ the following hold:

(i) There exists a positive constant $R_m \geqslant 2m$ such that if $|z_m| \geqslant R_m$, then

$$I_\lambda(t\tau_0 W_{m,z_m}) < \left(\frac{1}{2} - \frac{1}{p}\right)(\tau_0 \mathcal{S}_{\lambda,p})^{p/(p-2)}.$$

(ii) There exists a positive constant $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$, then

$$I_\lambda\big(t(1-\tau_0)U_{m,\varepsilon}\big) < \frac{1}{N}\big[(1-\tau_0)\mathbb{S}\big]^{N/2},$$

provided one of the following conditions holds:

(a) $N = 3$, $\alpha \geqslant 1$ and $p \in (4, 6)$;
(b) $N = 3$, $\alpha \geqslant 1$, $p = 4$ and η or ς large enough;
(c) $N = 3$, $\alpha \in (0, 1)$ and $p > 6 - 2\alpha$;
(d) $N = 3$, $\alpha \in (0, 1)$, $p = 6 - 2\alpha$ and ς large enough;
(e) $N \geqslant 4$ and $\alpha \geqslant 2$;
(f) $N \geqslant 4$, $\alpha \in (0, 2)$ and $p > \frac{2(N-\alpha)}{N-2}$;
(g) $N \geqslant 4$, $\alpha \in (0, 2)$, $p = \frac{2(N-\alpha)}{N-2}$ and ς large enough.

Case (i): We can find constants $t_2 > t_1 > 0$ such that for all $t \in [0, t_1] \cup [t_2, \infty)$, $m \in \mathbb{N}$ and $z_m \in \mathbb{R}^N$ with $|z_m| \geqslant 2m$,

$$I_\lambda(t\tau_0 W_{m,z_m}) < \left(\frac{1}{2} - \frac{1}{p}\right)(\tau_0 \mathcal{S}_{\lambda,p})^{p/(p-2)}.$$

In the following we may set $t \in [t_1, t_2]$. By the definition of W_{m,z_m},

$$I_\lambda(t\tau_0 W_{m,z_m}) = \frac{t^2\tau_0^2}{2} \int\limits_{\mathbb{R}^N} \big[\big|\nabla(\eta_m w_{z_m})\big|^2 + \big(V(x) - \lambda\big)(\eta_m w_{z_m})^2\big]\,dx$$

$$
\begin{aligned}
&-\frac{t^p\tau_0^p}{p}\int\limits_{\mathbb{R}^{\mathbb{N}}} a(x)|w_{m,z_m}|^p\,dx-\frac{t^{2^*}\tau_0^{2^*}}{2^*}\int\limits_{\mathbb{R}^{\mathbb{N}}} K(x)|w_{m,z_m}|^{2^*}\,dx\\
&\leqslant \frac{t^2\tau_0^2}{2}\int\limits_{\mathbb{R}^{\mathbb{N}}}\big[|\nabla\eta_m|^2|w_{z_m}|^2+2\eta_m w_{z_m}\nabla w_{z_m}\nabla\eta_m+\eta_m^2|\nabla w_{z_m}|^2\big]\,dx\\
&\quad+\frac{t^2\tau_0^2}{2}\int\limits_{\mathbb{R}^{\mathbb{N}}}\big(V(x)-\lambda\big)\eta_m^2 w_{z_m}^2\,dx-\frac{t^p\tau_0^p}{p}\int\limits_{\mathbb{R}^{\mathbb{N}}} a(x)\eta_m^p w_{z_m}^p\,dx-\frac{t^{2^*}\tau_0^{2^*}}{2^*}\int\limits_{\mathbb{R}^{\mathbb{N}}} K(x)\eta_m^{2^*} w_{z_m}^{2^*}\,dx\\
&\leqslant \frac{t^2\tau_0^2}{2}\int\limits_{\mathbb{R}^{\mathbb{N}}}\big[|\nabla w_{z_m}|^2+(1-\lambda)w_{z_m}^2\big]\,dx-\frac{t^p\tau_0^p}{p}\int\limits_{\mathbb{R}^{\mathbb{N}}} w_{z_m}^p\,dx-\frac{t^{2^*}\tau_0^{2^*}}{2^*}\int\limits_{\mathbb{R}^{\mathbb{N}}} K(x)\eta_m^{2^*} w_{z_m}^{2^*}\,dx\\
&\quad+\frac{t^2\tau_0^2}{2}\int\limits_{D_m}\big[2w_{z_m}^2+|\nabla w_{z_m}|^2\big]\,dx+\frac{t^2\tau_0^2}{2}\int\limits_{\mathbb{R}^{\mathbb{N}}}\big(\eta_m^2-1\big)|\nabla w_{z_m}|^2\,dx\\
&\quad+\frac{t^2\tau_0^2}{2}\int\limits_{\mathbb{R}^{\mathbb{N}}}\big[(V-\lambda)\eta_m^2-(1-\lambda)\big]w_{z_m}^2\,dx+\frac{t^p\tau_0^p}{p}\int\limits_{\mathbb{R}^{\mathbb{N}}}\big[1-a(x)\eta_m^p\big]w_{z_m}^p\,dx\\
&\leqslant \left(\frac{1}{2}-\frac{1}{p}\right)(\tau_0\mathcal{S}_{\lambda,p})^{p/(p-2)}-\frac{t_1^{2^*}\tau_0^{2^*}}{2^*}\int\limits_{\mathbb{R}^{\mathbb{N}}} K(x)\eta_m^{2^*} w_{z_m}^{2^*}\,dx\\
&\quad+\frac{t_2^2\tau_0^2}{2}\int\limits_{D_m}\big[2w_{z_m}^2+|\nabla w_{z_m}|^2\big]\,dx-\frac{t_1^2\tau_0^2}{2}\int\limits_{B_{m+1}}|\nabla w_{z_m}|^2\,dx\\
&\quad+\frac{\tau_0^2 t^2}{2}\int\limits_{\mathbb{R}^{\mathbb{N}}}\big[(V-\lambda)\eta_m^2-(1-\lambda)\big]w_{z_m}^2\,dx+\frac{\tau_0^p t^p}{p}\int\limits_{\mathbb{R}^{\mathbb{N}}}\big[1-a(x)\eta_m^p\big]w_{z_m}^p\,dx.
\end{aligned}
$$

By (K_1) and the definitions of η_m and w_{z_m}, there exists $r_0>0$ such that for every $m\in\mathbb{N}$ the following holds: there exists $R_m^1>0$ such that if $|z_m|\geqslant R_m^1$, then

$$\int\limits_{\mathbb{R}^{\mathbb{N}}} K(x)\eta_m^{2^*} w_{z_m}^{2^*}\,dx\geqslant r_0.$$

It follows from the definition of w_{z_m} that $w_{z_m}\rightharpoonup 0$ ($|z_m|\to\infty$) in $H^1(\mathbb{R}^N)$. By D_m and B_{m+1} are bounded domain, for $|z_m|\to\infty$ we have

$$\int\limits_{D_m}\big[2w_{z_m}^2+|\nabla w_{z_m}|^2\big]\,dx\to 0$$

and

$$\int\limits_{B_{m+1}}|\nabla w_{z_m}|^2\,dx\to 0.$$

By

$$(V-\lambda)\eta_m^2-(1-\lambda)\to 0,\quad |x|\to\infty,$$

and

$$1-a(x)\eta_m^p\to 0,\quad |x|\to\infty,$$

for $|z_m|\to\infty$ we have

$$\int_{\mathbb{R}^N} \left[(V-\lambda)\eta_m^2 - (1-\lambda)\right] w_{z_m}^2\, dx \to 0$$

and

$$\int_{\mathbb{R}^N} \left[1 - a(x)\eta_m^p\right] w_{z_m}^p\, dx \to 0.$$

Thus, we obtain Case (i).

Case (ii): By (5.1), (K_1) and the continuity of I_λ, There exist constants $\hat{\varepsilon} > 0$ and $t_4 > t_3 > 0$ such that for all $\varepsilon \in (0, \hat{\varepsilon})$, $t \in [0, t_3] \cup [t_4, \infty)$ and $m \in \mathbb{N}$,

$$I_\lambda\big(t(1-\tau_0)U_{m,\varepsilon}\big) < \frac{1}{N}\left[(1-\tau_0)\mathbb{S}\right]^{N/2}.$$

In the following we may set $t \in [t_3, t_4]$ and $\varepsilon \in (0, \hat{\varepsilon})$. By the definition of $U_{m,\varepsilon}$,

$$\begin{aligned} I_\lambda\big(t(1-\tau_0)U_{m,\varepsilon}\big) &= \frac{t^2(1-\tau_0)^2}{2}\int_{\mathbb{R}^N} \left[|\nabla U_{m,\varepsilon}|^2 + \big(V(x)-\lambda\big)U_{m,\varepsilon}^2\right] dx \\ &\quad - \frac{t^p(1-\tau_0)^p}{p}\int_{\mathbb{R}^N} a(x)U_{m,\varepsilon}^p\, dx - \frac{t^{2^*}(1-\tau_0)^{2^*}}{2^*}\int_{\mathbb{R}^N} K(x)U_{m,\varepsilon}^{2^*}\, dx \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where

$$I_1 := \frac{t^2(1-\tau_0)^2}{2}\int_{\mathbb{R}^N} |\nabla U_{m,\varepsilon}|^2\, dx - \frac{t^{2^*}(1-\tau_0)^{2^*}}{2^*}\int_{\mathbb{R}^N} K(x)U_{m,\varepsilon}^{2^*}\, dx,$$

$$I_2 := \frac{t^2(1-\tau_0)^2}{2}\int_{\mathbb{R}^N} \big(V(x)-\lambda\big)U_{m,\varepsilon}^2\, dx$$

and

$$I_3 := -\frac{t^p(1-\tau_0)^p}{p}\int_{\mathbb{R}^N} a(x)U_{m,\varepsilon}^p\, dx.$$

By (5.1) and Lemma 5.1,

$$\begin{aligned} I_1 &= \frac{t^2}{2}\Big(\left[(1-\tau_0)\mathbb{S}\right]^{N/2} + O\big(\varepsilon^{N-2}\big)\Big) - \frac{t^{2^*}}{2^*}\Big(\left[(1-\tau_0)\mathbb{S}\right]^{N/2} + O\big(h(\varepsilon)\big)\Big) \\ &\leqslant \frac{1}{N}\left[(1-\tau_0)\mathbb{S}\right]^{N/2} + O\big(\varepsilon^\beta\big), \end{aligned} \tag{6.2}$$

where $\beta = \min\{\alpha, N-2\}$. From $m > M_0$ and (5.2),

$$\begin{aligned} I_2 &\leqslant -t_3\frac{\eta}{2}\int_{\mathbb{R}^N} u_\varepsilon^2\, dx \\ &\leqslant \begin{cases} -b_0\varepsilon^2 + O(\varepsilon^{N-2}) & \text{if } N \geqslant 5, \\ -b_0\varepsilon^2|\log\varepsilon| + O(\varepsilon^2) & \text{if } N = 4, \\ -b_0\varepsilon + O(\varepsilon^2) & \text{if } N = 3, \end{cases} \end{aligned} \tag{6.3}$$

where $b_0 = t_3 b\eta/2 > 0$.

In the following we estimate I_3. Taking $\varepsilon < \frac{1}{2m}$,

$$\int_{B_{\frac{1}{2m}}} U_{m,\varepsilon}^p\,dx = \int_{B_{\frac{1}{2m}}} u_\varepsilon^p\,dx = \int_{B_\varepsilon} u_\varepsilon^p\,dx + \int_{B_{\frac{1}{2m}}\setminus B_\varepsilon} u_\varepsilon^p\,dx.$$

Noting that $2 < p < 2N/(N-2)$,

$$\begin{aligned}\int_{B_\varepsilon} u_\varepsilon^p\,dx &= \int_{B_\varepsilon} \frac{[N(N-2)]^{p(N-2)/4}\varepsilon^{p(N-2)/2}}{(\varepsilon^2+|x|^2)^{p(N-2)/2}}\,dx \\ &\geqslant 2^{-p(N-2)/2}\big[N(N-2)\big]^{p(N-2)/4}\varepsilon^{-p(N-2)/2}\int_{B_\varepsilon} dx \\ &= 2^{-p(N-2)/2}\big[N(N-2)\big]^{p(N-2)/4} w_N \varepsilon^{N-p(N-2)/2}\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}\int_{B_{\frac{1}{2m}}\setminus B_\varepsilon} u_\varepsilon^p\,dx &= \int_{B_{\frac{1}{2m}}\setminus B_\varepsilon} \frac{[N(N-2)]^{p(N-2)/4}\varepsilon^{p(N-2)/2}}{(\varepsilon^2+|x|^2)^{p(N-2)/2}}\,dx \\ &\geqslant 2^{-p(N-2)/2}\big[N(N-2)\big]^{p(N-2)/4} w_N \varepsilon^{p(N-2)/2}\int_\varepsilon^{\frac{1}{2m}} \rho^{-p(N-2)}\rho^{N-1}\,d\rho \\ &= \big[N(N-2)\big]^{p(N-2)/4} w_N \varepsilon^{p(N-2)/2}\int_\varepsilon^{\frac{1}{2m}} \rho^{N-p(N-2)-1}\,d\rho \\ &\geqslant \begin{cases} c_{15}[N(N-2)]^{p(N-2)/4} w_N \varepsilon^{p(N-2)/2}, & p < \frac{N}{N-2}, \\ c_{15}[N(N-2)]^{p(N-2)/4} w_N \varepsilon^{p(N-2)/2}|\ln\varepsilon|, & p = \frac{N}{N-2}, \\ c_{15}[N(N-2)]^{p(N-2)/4} w_N \varepsilon^{N-p(N-2)/2}, & p > \frac{N}{N-2}, \end{cases}\end{aligned} \tag{6.5}$$

where w_N is the volume of unit ball in $\mathbb{R}^N$ and c_{15} is a positive constant.

Combining (6.4) and (6.5), we obtain

$$I_3 \leqslant \begin{cases} \left.\begin{array}{ll} -t_3 c_{16}\varsigma\varepsilon^{p/2}, & 2 < p < 3 \\ -t_3 c_{16}\varsigma\varepsilon^{3/2}|\ln\varepsilon|, & p = 3 \\ -t_3 c_{16}\varsigma\varepsilon^{3-p/2}, & 3 < p < 6 \end{array}\right\} & N = 3, \\ -t_3 c_{16}\varsigma\varepsilon^{N-p(N-2)/2}, & N \geqslant 4, \end{cases} \tag{6.6}$$

where c_{16} is a positive constant.

Hence, by (6.2), (6.3) and (6.6) for $N = 3$, $\alpha \geqslant 1$ and $p \in (4,6)$,

$$I_\lambda\big(t(1-\tau_0)U_{m,\varepsilon}\big) \leqslant \frac{1}{N}\big[(1-\tau_0)\mathbb{S}\big]^{N/2} + O(\varepsilon) - b_0\varepsilon - t_3 c_{16}\varsigma\varepsilon^{3-p/2} + O\big(\varepsilon^2\big). \tag{a}$$

For $N = 3$, $\alpha \geqslant 1$ and $p = 4$,

$$I_\lambda\big(t(1-\tau_0)U_{m,\varepsilon}\big) \leqslant \frac{1}{N}\big[(1-\tau_0)\mathbb{S}\big]^{N/2} + O(\varepsilon) - b_0\varepsilon - t_3 c_{16}\varsigma\varepsilon + O\big(\varepsilon^2\big). \tag{b}$$

For $N = 3$, $\alpha \in (0,1)$ and $p > 6 - 2\alpha$,

$$I_\lambda\big(t(1-\tau_0)U_{m,\varepsilon}\big) \leqslant \frac{1}{N}\big[(1-\tau_0)\mathbb{S}\big]^{N/2} + C_\alpha\varepsilon^\alpha - t_3 c_{16}\varsigma\varepsilon^{3-p/2}. \tag{c}$$

For $N=3$, $\alpha\in(0,1)$ and $p=6-2\alpha$,

$$I_\lambda\big(t(1-\tau_0)U_{m,\varepsilon}\big)\leqslant\frac{1}{N}\big[(1-\tau_0)\mathbb{S}\big]^{N/2}+C_\alpha\varepsilon^\alpha-t_3c_{16}\varsigma\varepsilon^{3-p/2}. \tag{d}$$

For $N\geqslant 4$ and $\alpha\geqslant 2$,

$$I_\lambda\big(t(1-\tau_0)U_{m,\varepsilon}\big)\leqslant\frac{1}{N}\big[(1-\tau_0)\mathbb{S}\big]^{N/2}+O\big(\varepsilon^\beta\big)-t_3c_{16}\varsigma\varepsilon^{N-p(N-2)/2}. \tag{e}$$

For $N\geqslant 4$, $\alpha\in(0,2)$ and $p>\frac{2(N-\alpha)}{N-2}$,

$$I_\lambda\big(t(1-\tau_0)U_{m,\varepsilon}\big)\leqslant\frac{1}{N}\big[(1-\tau_0)\mathbb{S}\big]^{N/2}+C_\alpha\varepsilon^\alpha-t_3c_{16}\varsigma\varepsilon^{N-p(N-2)/2}. \tag{f}$$

For $N\geqslant 4$, $\alpha\in(0,2)$ and $p=\frac{2(N-\alpha)}{N-2}$,

$$I_\lambda\big(t(1-\tau_0)U_{m,\varepsilon}\big)\leqslant\frac{1}{N}\big[(1-\tau_0)\mathbb{S}\big]^{N/2}+C_\alpha\varepsilon^\alpha-t_3c_{16}\varsigma\varepsilon^{N-p(N-2)/2}. \tag{g}$$

Choosing ε small enough in (a), (c), (e) and (f); ε small enough and η or ς large enough in (b); ε small enough and ς large enough in (d) and (g), we get

$$I_\lambda\big(t(1-\tau_0)U_{m,\varepsilon}\big)<\frac{1}{N}\big[(1-\tau_0)\mathbb{S}\big]^{N/2}. \tag{6.7}$$

Thus, we obtain Case (ii).

Combining Case (i) and Case (ii), we complete the proof. □

Proof of Theorem 1.1. In order to apply Theorem A.1, we set $N=S_3(\rho)$ and $M=Q_m^{\varepsilon,z_m,R_1}$, where $S_3(\rho)$ and $Q_m^{\varepsilon,z_m,R_1}$ appear in Lemma 5.3. Noting that $\lambda\in(\lambda_i-\delta_1,\lambda_i)$, $\delta_1>0$, $m\geqslant M_0$, $z_m\geqslant 2m$ and $R_1>\rho$. Take $m>\bar{m}$ and $R_1>R_m$, where $\bar{m}$ and R_m appear in Lemma 6.1, under one of conditions: (a)–(g) we have c (see in (A.1)) belongs to $(0,\mathfrak{S})$. Then by Lemma 3.2, the $(\mathrm{PS})_c$-sequence $\{u_n\}$ of I_λ has a subsequence which converges strongly in $H^1(\mathbb{R}^N)$. Going if necessary to a subsequence, we can assume that $u_n\to u$ in $H^1(\mathbb{R}^N)$. Clearly, $I_\lambda(u)=c$ and $I'_\lambda(u)=0$. Then Problem $(\mathscr{P}_\lambda)$ has at least one nontrivial solution $u\in H^1(\mathbb{R}^N)$. From (5.3), we imply that $I_\lambda(u)\geqslant c_{14}\rho^2$, where $\rho>0$ and $c_{14}>0$ are constants independent of λ.

Take $\delta_2>0$ and find ε_0 as in Lemma 4.4. Fix $0<\varepsilon'<\varepsilon''<\varepsilon_0$, where we request $\varepsilon''<c_{14}\rho^2$. According to Lemmas 5.2 and 5.3 there exists $\delta_0\leqslant\min\{\delta_1,\delta_2\}$ such that for $\lambda\in(\lambda_i-\delta_0,\lambda_i)$, $\sup\{I_\lambda(X)\}<\varepsilon''$. Then by Lemma 4.4, the condition $(\nabla)(I_\lambda,X_1\oplus X_3,\varepsilon',\varepsilon'')$ holds. Combining Lemma 4.1, Lemma 3.2 and Theorem A.2, there are two nontrivial solutions such that $I_\lambda(u_i)\in[\varepsilon',\varepsilon'']$, $i=1,2$. Noting that $0<\varepsilon'\leqslant I_\lambda(u_i)\leqslant\varepsilon''<c_{14}\rho^2\leqslant I_\lambda(u)$, $i=1,2$.

So Theorem 1.1 is proved. □

Appendix A

Let E be a Banach space.

Theorem A.1 *(Linking theorem). Let $E=X\oplus W$ with $\dim X<\infty$. Let $R>\rho>0$ and let $z\in W$ be such that $\|z\|=\rho$. Define*

$$M:=\big\{u=y+tz\colon\ \|u\|\leqslant R,\ t\geqslant 0,\ y\in X\big\},$$
$$M_0:=\big\{u=y+tz\colon\ y\in X,\ \|u\|=R\ \textit{and}\ t\geqslant 0\ \textit{or}\ \|u\|\leqslant R\ \textit{and}\ t=0\big\},$$
$$N:=\big\{u\in W\colon\ \|u\|=\rho\big\}.$$

Let $\varphi\in C^1(E,\mathbb{R})$ be such that

$$b:=\inf_N\varphi>a:=\max_{M_0}\varphi.$$

If φ satisfies the $(\mathrm{PS})_c$ *condition with*

$$c := \inf_{h\in\Gamma}\max_{u\in M}\varphi\big(h(u)\big), \tag{A.1}$$

$$\Gamma := \big\{h\in C(M,E)\colon\ h|_{M_0}=\mathrm{id}\big\},$$

then c is a critical value of φ.

Definition. Let X be a closed subspace of H, which is a Hilbert space, $a,b\in\mathbb{R}\cup\{-\infty,+\infty\}$. We say that a C^1 function $f: H\to\mathbb{R}$ verifies the condition $(\nabla)(f,X,a,b)$ if there exists $\gamma>0$ such that

$$\inf\big\{\big\|P_X\nabla f(u)\big\|\ \big|\ a\leqslant f(u)\leqslant b,\ \mathrm{dist}(u,X)\leqslant\gamma\big\}>0,$$

where $P_X: H\to X$ denotes the orthogonal projection of H onto X.

The (∇)-theorem in the following:

Theorem A.2 *(Sphere–torus linking with mixed type assumptions). Let H be a Hilbert space and X_i, $i=1,2,3$, be three subspaces of H such that $H=X_1\oplus X_2\oplus X_3$ and $\dim X_i<\infty$ for $i=1,2$. Let $\varphi: H\to\mathbb{R}$ be a $C^{1,1}$ function. Denote by P_i the orthogonal projection of H onto X_i. Let ρ,ρ',ρ'',ρ_1 be such that $\rho_1>0$, $0\leqslant\rho'<\rho<\rho''$. Assume*

$$a'=\sup_{u\in T}\varphi(u)<\inf_{S_{23}(\rho)}\varphi(w)=a'',$$

where

$$T=\partial_{X_1\oplus X_2}\Delta\quad\textit{and}\quad \Delta=\big\{u\in X_1\oplus X_2\ \big|\ \rho'\leqslant\|P_2u\|\leqslant\rho'',\ \|P_1u\|\leqslant\rho_1\big\},$$
$$S_{23}(\rho)=\big\{u\in X_2\oplus X_3\ \big|\ \|u\|=\rho\big\}.$$

Let a and b be such that $a'<a<a''$ and $b>\sup\varphi(\Delta)$ and the assumption $(\nabla)(\varphi,X_1\oplus X_2,a,b)$ holds. Finally assume that $(\mathrm{PS})_c$-condition holds at any c in $[a,b]$. Then φ has at least two critical points in $\varphi^{-1}([a,b])$. If furthermore,

$$-\infty<\inf_{B_{23}(\rho)}\varphi(w)=a_1,$$

where $B_{23}(\rho)=\{u\in X_2\oplus X_3\mid\|u\|\leqslant\rho\}$, and $(\mathrm{PS})_c$-condition holds at any $c\in[a_1,a]$, then f has another critical level in $[a_1,a]$.

Appendix B

B.1. Proof of Lemma 4.1

By $0<\lambda<\lambda_i$, for $z\in X_2\oplus X_3$ we have

$$\int_{\mathbb{R}^{\mathbb{N}}}\big[|\nabla z|^2+\big(V(x)-\lambda\big)z^2\big]\,dx=\|z\|^2-\frac{\lambda}{\lambda_i}\lambda_i\int_{\mathbb{R}^{\mathbb{N}}}z^2\,dx\geqslant\left(1-\frac{\lambda}{\lambda_i}\right)\|z\|^2.$$

Then

$$\begin{aligned}I_\lambda(z)&=\frac12\int_{\mathbb{R}^{\mathbb{N}}}\big[|\nabla z|+\big(V(x)-\lambda\big)z^2\big]\,dx-\frac1p\int_{\mathbb{R}^{\mathbb{N}}}a(x)|z|^p\,dx-\frac{1}{2^*}\int_{\mathbb{R}^{\mathbb{N}}}K(x)|z|^{2^*}\,dx\\&\geqslant\left(1-\frac{\lambda}{\lambda_i}\right)\|z\|^2-c_1\|z\|^p-c_2\|z\|^{2^*},\end{aligned}$$

where c_1 and c_2 are positive constants. So there exists a constant $\rho>0$ such that

$$\inf I_\lambda\big(S_{23}(\rho)\big)>0.$$

From the definition of X_1, for $w \in X_1$,

$$I_\lambda(w) = \frac{1}{2}\int_{\mathbb{R}^{\mathbb{N}}} \left[|\nabla w| + \big(V(x)-\lambda\big)w^2\right] dx - \frac{1}{p}\int_{\mathbb{R}^{\mathbb{N}}} a(x)|w|^p\, dx - \frac{1}{2^*}\int_{\mathbb{R}^{\mathbb{N}}} K(x)|w|^{2^*} dx$$
$$\leqslant (\lambda_{i-1}-\lambda)\int_{\mathbb{R}^{\mathbb{N}}} w^2\, dx - \frac{1}{p}\int_{\mathbb{R}^{\mathbb{N}}} a(x)|w|^p\, dx$$
$$\leqslant 0.$$

To conclude the proof it is enough to show that

$$\lim_{\|v\|\to\infty,\, v\in X_1\oplus X_2} I_\lambda(u) = -\infty.$$

In fact, for $v \in X_1 \oplus X_2$,

$$I_\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^{\mathbb{N}}} \left[|\nabla v|^2 + \big(V(x)-\lambda\big)v^2\right] dx - \frac{1}{p}\int_{\mathbb{R}^{\mathbb{N}}} a(x)|v|^p\, dx - \frac{1}{2^*}\int_{\mathbb{R}^{\mathbb{N}}} K(x)|v|^{2^*} dx$$
$$\leqslant \frac{(\lambda_i-\lambda)}{2}\int_{\mathbb{R}^{\mathbb{N}}} v^2\, dx - \frac{1}{p}\int_{\mathbb{R}^{\mathbb{N}}} a(x)|v|^p\, dx.$$

Since all norms in $X_1 \oplus X_2$ are equivalent, the results follow.

B.2. Proof of Lemma 4.2

By contradiction, let us suppose that there exist $\lambda_n \in [0, \lambda_i]$ and $u_n \in X_1 \oplus X_3 \backslash \{0\}$ such that

$$I_{\lambda_n}(u_n) = \frac{1}{2}\int_{\mathbb{R}^{\mathbb{N}}} \left[|\nabla u_n|^2 + \big(V(x)-\lambda_n\big)u_n^2\right] dx - \frac{1}{p}\int_{\mathbb{R}^{\mathbb{N}}} a(x)|u_n|^p\, dx - \frac{1}{2^*}\int_{\mathbb{R}^{\mathbb{N}}} K(x)|u_n|^{2^*} dx \to 0 \tag{B.1}$$

and for all $z \in X_1 \oplus X_3$,

$$\int_{\mathbb{R}^{\mathbb{N}}} \left[\nabla u_n \cdot \nabla z + V(x)u_n z - \lambda_n u_n z\right] dx - \int_{\mathbb{R}^{\mathbb{N}}} a(x)|u_n|^{p-2}u_n z\, dx - \int_{\mathbb{R}^{\mathbb{N}}} K(x)|u_n|^{2^*-2}u_n z\, dx = 0. \tag{B.2}$$

Choose $z = u_n$ in (B.2). Then by (B.1), we have

$$\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^{\mathbb{N}}} a(x)|u_n|^p\, dx + \left(\frac{1}{2}-\frac{1}{2^*}\right)\int_{\mathbb{R}^{\mathbb{N}}} K(x)|u_n|^{2^*} dx \to 0 \quad (n\to\infty).$$

Thus

$$\int_{\mathbb{R}^{\mathbb{N}}} a(x)|u_n|^p\, dx, \qquad \int_{\mathbb{R}^{\mathbb{N}}} K(x)|u_n|^{2^*} dx \to 0 \quad (n\to\infty). \tag{B.3}$$

Choose $z = w_n - v_n$ in (B.2), where $v_n \in X_1$ and $w_n \in X_3$ such that $u_n = v_n + w_n$. Then we get

$$\int_{\mathbb{R}^{\mathbb{N}}} \left[|\nabla w_n|^2 + \big(V(x)-\lambda_n\big)w_n^2\right] dx - \int_{\mathbb{R}^{\mathbb{N}}} \left[|\nabla v_n|^2 + \big(V(x)-\lambda_n\big)v_n^2\right] dx$$
$$= \int_{\mathbb{R}^{\mathbb{N}}} a(x)|v_n+w_n|^{p-2}(v_n+w_n)(w_n-v_n)\, dx + \int_{\mathbb{R}^{\mathbb{N}}} K(x)|v_n+w_n|^{2^*-2}(v_n+w_n)(w_n-v_n)\, dx. \tag{B.4}$$

About the first line in (B.4),

$$\begin{aligned}
&\int_{\mathbb{R}^N} \big[|\nabla w_n|^2 + \big(V(x)-\lambda_n\big)w_n^2\big]\,dx - \int_{\mathbb{R}^N} \big[|\nabla v_n|^2 + \big(V(x)-\lambda_n\big)v_n^2\big]\,dx \\
&\quad \geqslant \int_{\mathbb{R}^N} \big[|\nabla w_n|^2 + V(x)w_n^2\big]\,dx - \frac{\lambda_n}{\lambda_{i+1}}\lambda_{i+1}\int_{\mathbb{R}^N} w_n^2\,dx - \int_{\mathbb{R}^N} \big[|\nabla v_n|^2 + V(x)v_n^2\big]\,dx \\
&\quad \geqslant (1-\lambda_n/\lambda_{i+1})\|w_n\|^2 + \|v_n\|^2 \\
&\quad \geqslant (1-\lambda_n/\lambda_{i+1})\|u_n\|^2.
\end{aligned} \tag{B.5}$$

The integrals on the right side of (B.4) can be estimated in the following way. By the Hölder and Sobolev's inequalities,

$$\begin{aligned}
\int_{\mathbb{R}^N} a(x)|v_n+w_n|^{p-2}(v_n+w_n)(w_n-v_n)\,dx &\leqslant \int_{\mathbb{R}^N} a(x)|v_n+w_n|^{p-1}|w_n-v_n|\,dx \\
&\leqslant \Big[\int_{\mathbb{R}^N} a(x)|u_n|^p\,dx\Big]^{(p-1)/p}\Big[\int_{\mathbb{R}^N} a(x)|w_n-v_n|^p\,dx\Big]^{1/p} \\
&\leqslant \Big[\int_{\mathbb{R}^N} a(x)|u_n|^p\,dx\Big]^{(p-1)/p} c_3\|u_n\| \\
&\leqslant c_4\|u_n\|^p
\end{aligned} \tag{B.6}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^N} K(x)|v_n+w_n|^{2^*-2}(v_n+w_n)(w_n-v_n)\,dx &\leqslant \int_{\mathbb{R}^N} K(x)|v_n+w_n|^{2^*-1}|w_n-v_n|\,dx \\
&\leqslant \Big[\int_{\mathbb{R}^N} K(x)|u_n|^{2^*}\,dx\Big]^{(2^*-1)/2^*}\Big[\int_{\mathbb{R}^N} K(x)|w_n-v_n|^{2^*}\,dx\Big]^{1/2^*} \\
&\leqslant \Big[\int_{\mathbb{R}^N} K(x)|u_n|^{2^*}\,dx\Big]^{(2^*-1)/2^*} c_5\|u_n\| \\
&\leqslant c_6\|u_n\|^{2^*},
\end{aligned} \tag{B.7}$$

where c_3, c_4, c_5 and c_6 are positive constants. Combining (B.5), (B.6) and (B.7),

$$\begin{aligned}
c_4\|u_n\|^p + c_6\|u_n\|^{2^*} &\geqslant \Big[\int_{\mathbb{R}^N} a(x)|u_n|^p\,dx\Big]^{(p-1)/p} c_3\|u_n\| + \Big[\int_{\mathbb{R}^N} K(x)|u_n|^{2^*}\,dx\Big]^{(2^*-1)/2^*} c_5\|u_n\| \\
&\geqslant (1-\lambda_n/\lambda_{j+1})\|u_n\|^2.
\end{aligned}$$

So $\|u_n\| \to 0$ $(n\to\infty)$ since (B.3) and $\|u_n\| \geqslant c_7 > 0$ since $p, 2^* > 2$, where c_7 is a positive constant. This is a self-contradictory.

B.3. Proof of Lemma 4.3

By contradiction, let us assume that $\|u_n\| \to \infty$. Define $v_n = u_n/\|u_n\|$. Then there is u in $H^1(\mathbb{R}^N)$ such that $v_n \rightharpoonup u$ (passing to a subsequence, if necessary). From the definitions of P_2 and P_{13}, we have

$$\begin{aligned}\langle P_{13}I'_\lambda(u_n),u_n\rangle &= \langle I'_\lambda(u_n),u_n\rangle - \langle P_2 I'_\lambda(u_n),u_n\rangle\\ &= 2I_\lambda(u_n) + \left(\frac{2}{p}-1\right)\int_{\mathbb{R}^{\mathbb{N}}} a(x)|u_n|^p\,dx - \frac{2}{N}\int_{\mathbb{R}^{\mathbb{N}}} K(x)|u_n|^{2^*}\,dx\\ &\quad - \int_{\mathbb{R}^{\mathbb{N}}}\left[\left|\nabla(P_2u_n)\right|^2 + \left(V(x)-\lambda\right)(P_2u_n)^2\right]dx\\ &\quad + \int_{\mathbb{R}^{\mathbb{N}}} a(x)|u_n|^{p-2}u_nP_2u_n\,dx + \int_{\mathbb{R}^{\mathbb{N}}} K(x)|u_n|^{2^*-2}u_nP_2u_n\,dx.\end{aligned}$$

Then by Sobolev inequality,

$$\begin{aligned}&\langle P_{13}I'_\lambda(u_n),u_n\rangle + \left(1-\frac{2}{p}\right)\int_{\mathbb{R}^{\mathbb{N}}} a(x)|u_n|^p\,dx + \frac{2}{N}\int_{\mathbb{R}^{\mathbb{N}}} K(x)|u_n|^{2^*}\,dx\\ &\leqslant 2I_\lambda(u_n) - \int_{\mathbb{R}^{\mathbb{N}}}\left[|\nabla P_2u_n|^2 + \left(V(x)-\lambda\right)(P_2u_n)^2\right]dx\\ &\quad + \left(\int_{\mathbb{R}^{\mathbb{N}}} a(x)|u_n|^p\,dx\right)^{(p-1)/p}\left(\int_{\mathbb{R}^{\mathbb{N}}} a(x)|P_2u_n|^p\,dx\right)^{1/p}\\ &\quad + \left(\int_{\mathbb{R}^{\mathbb{N}}} K(x)|u_n|^{2^*}\,dx\right)^{(2^*-1)/2^*}\left(\int_{\mathbb{R}^{\mathbb{N}}} K(x)|P_2u_n|^{2^*}\,dx\right)^{1/2^*}\\ &\leqslant 2I_\lambda(u_n) - \int_{\mathbb{R}^{\mathbb{N}}}\left[|\nabla P_2u_n|^2 + \left(V(x)-\lambda\right)(P_2u_n)^2\right]dx\\ &\quad + c_8\|u_n\|^{p-1}\|P_2u_n\| + c_8\|u_n\|^{2^*-1}\|P_2u_n\|,\end{aligned} \tag{B.8}$$

where c_8 is a positive constant. Using $P_2u_n \to 0$ and dividing by $\|u_n\|^{2^*-1}$, we conclude

$$\lim_{n\to\infty}\frac{\int_{\mathbb{R}^{\mathbb{N}}} K(x)|u_n|^{2^*}\,dx}{\|u_n\|^{2^*-1}} = 0.$$

So

$$\lim_{n\to\infty}\int_{\mathbb{R}^{\mathbb{N}}} K(x)|v_n|^{2^*}\,dx\cdot\|u_n\| = 0.$$

By $\|u_n\|\to\infty$, we have $\lim_{n\to\infty}\int_{\mathbb{R}^{\mathbb{N}}} K(x)|v_n|^{2^*}\,dx = 0$. It is easy to see that $v=0$, since $v_n \rightharpoonup v$.

Certainly, $\tau_n = \|u_n\|_\lambda \to \infty$ and $z_n = \frac{u_n}{\|u_n\|_\lambda} \rightharpoonup 0$. So

$$\frac{I_\lambda(u_n)}{\tau_n^2} = \frac{1}{2}\int_{\mathbb{R}^{\mathbb{N}}}|\nabla z_n|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^{\mathbb{N}}}\left(V(x)-\lambda\right)z_n^2\,dx - \frac{1}{p}\frac{\int_{\mathbb{R}^{\mathbb{N}}}|u_n|^p\,dx}{\tau_n^2} - \frac{1}{2^*}\frac{\int_{\mathbb{R}^{\mathbb{N}}}|u_n|^{2^*}\,dx}{\tau_n^2}.$$

Since $z_n \rightharpoonup 0$ and $\lim_{|x|\to\infty}V(x) = v_\infty = 1$, we obtain

$$\int_{\mathbb{R}^{\mathbb{N}}} V(x)z_n^2\,dx = \int_{\mathbb{R}^{\mathbb{N}}} z_n^2\,dx + o(1).$$

Therefore, for n large enough,

$$\lim_{n\to\infty}\left(\frac{1}{p}\frac{\int_{\mathbb{R}^{\mathbb{N}}} a(x)|u_n|^p\,dx}{\tau_n^2} + \frac{1}{2^*}\frac{\int_{\mathbb{R}^{\mathbb{N}}} K(x)|u_n|^{2^*}\,dx}{\tau_n^2}\right) = \frac{1}{2}.$$

Thus,

$$c_9 \leqslant \lim_{n\to\infty} \frac{\int_{\mathbb{R}^{\mathbb{N}}} a(x)|u_n|^p\,dx}{\tau_n^2}, \qquad \lim_{n\to\infty} \frac{\int_{\mathbb{R}^{\mathbb{N}}} K(x)|u_n|^{2^*}\,dx}{\tau_n^2} \leqslant c_{10},$$

where c_9 and c_{10} are positive constants. Then from (B.8), for n large enough we get

$$c_{11}\|u_n\|^2 \leqslant c_{12} + \|P_2u_n\| + c_{13}\|u_n\|^{2(p-1)/p}\|P_2u_n\| + c_{13}\|u_n\|^{2(2^*-1)/2^*}\|P_2u_n\|,$$

where c_{11}, c_{12} and c_{13} are positive constants. So $\|u_n\|$ is bounded and a contradiction arises.

B.4. Proof of Lemma 4.4

By contradiction, let us suppose that $\exists\bar\delta > 0$ such that $\forall\varepsilon_0 > 0$ if $\lambda \in (\lambda_i - \bar\delta, \lambda_i)$ and $\varepsilon', \varepsilon'' \in (0, \varepsilon_0)$, then the condition $(\nabla)(I_\lambda, X_1 \oplus X_3, \varepsilon', \varepsilon'')$ does not hold. Take $\varepsilon_0 = \min\{\varepsilon_1, \mathfrak{S}\}$, where ε_1 and $\mathfrak{S}$ are given in Lemma 4.2 and (3.2), respectively. Then there exists $\{u_n\}_1^\infty \subset H^1(\mathbb{R}^N)$ such that $d(u_n, X_1 \oplus X_3) \to 0$, $I_\lambda(u_n) \in [\varepsilon', \varepsilon'']$ and $P_{13}I'_\lambda(u_n) \to 0$. By Lemma 4.3, $\{u_n\}_1^\infty$ is a bounded sequence. So we may assume, going if necessary to a subsequence,

$$\begin{aligned} &u_n \rightharpoonup v \quad \text{in } H^1(\mathbb{R}^N),\\ &u_n \to v \quad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^N). \end{aligned} \tag{B.9}$$

From $P_{13}I'_\lambda(u_n) \to 0$, we have

$$\begin{aligned} &\int_{\mathbb{R}^{\mathbb{N}}} \big[\nabla u_n \nabla(P_{13}\varphi) + \big(V(x)-\lambda\big)u_n P_{13}\varphi\big]\,dx\\ &\quad = \int_{\mathbb{R}^{\mathbb{N}}} a(x)|u_n|^{p-2}u_n P_{13}\varphi\,dx + \int_{\mathbb{R}^{\mathbb{N}}} K(x)|u_n|^{2^*-2}u_n P_{13}\varphi\,dx + o(1)\|\varphi\|, \end{aligned}$$

where $\varphi \in H^1(\mathbb{R}^N)$. Furthermore, by $P_2u_n \to 0$ we conclude that $v \in X_1 \oplus X_3$. Then from (B.9) we have

$$\int_{\mathbb{R}^{\mathbb{N}}} \big[\nabla v \nabla(P_{13}\varphi) + \big(V(x)-\lambda\big)v P_{13}\varphi\big]\,dx = \int_{\mathbb{R}^{\mathbb{N}}} a(x)|v|^{p-2}v P_{13}\varphi\,dx + \int_{\mathbb{R}^{\mathbb{N}}} K(x)|v|^{2^*-2}v P_{13}\varphi\,dx.$$

Take $\varphi = v$, we get

$$\int_{\mathbb{R}^{\mathbb{N}}} \big[|\nabla v|^2 + \big(V(x)-\lambda\big)v^2\big]\,dx = \int_{\mathbb{R}^{\mathbb{N}}} a(x)|v|^p\,dx + \int_{\mathbb{R}^{\mathbb{N}}} K(x)|v|^{2^*}\,dx. \tag{B.10}$$

Thus,

$$\begin{aligned} I_\lambda(v) &= \frac12 \int_{\mathbb{R}^{\mathbb{N}}} \big[|\nabla v|^2 + \big(V(x)-\lambda\big)v^2\big]\,dx - \frac1p \int_{\mathbb{R}^{\mathbb{N}}} a(x)|v|^p\,dx - \frac{1}{2^*}\int_{\mathbb{R}^{\mathbb{N}}} K(x)|v|^{2^*}\,dx\\ &= \left(\frac12 - \frac1p\right)\int_{\mathbb{R}^{\mathbb{N}}} a(x)|v|^p\,dx + \left(\frac12 - \frac{1}{2^*}\right)\int_{\mathbb{R}^{\mathbb{N}}} K(x)|v|^{2^*}\,dx \geqslant 0. \end{aligned} \tag{B.11}$$

Let $v_n = u_n - v$. By Lemma 1.32 of [18], (V_1) and (A_1),

$$\begin{aligned} \int_{\mathbb{R}^{\mathbb{N}}} \big[|\nabla u_n|^2 + \big(V(x)-\lambda\big)u_n^2\big]\,dx &= \int_{\mathbb{R}^{\mathbb{N}}} \big[|\nabla v_n|^2 + \big(V(x)-\lambda\big)v_n^2\big]\,dx + \int_{\mathbb{R}^{\mathbb{N}}} \big[|\nabla v|^2 + \big(V(x)-\lambda\big)v^2\big]\,dx + o(1)\\ &= \int_{\mathbb{R}^{\mathbb{N}}} \big[|\nabla v_n|^2 + (1-\lambda)v_n^2\big]\,dx + \int_{\mathbb{R}^{\mathbb{N}}} \big[|\nabla v|^2 + \big(V(x)-\lambda\big)v^2\big]\,dx + o(1), \end{aligned}$$

$$\int_{\mathbb{R}^{\mathbb{N}}} a(x)|u_n|^p\,dx = \int_{\mathbb{R}^{\mathbb{N}}} |v_n|^p\,dx + \int_{\mathbb{R}^{\mathbb{N}}} a(x)|v|^p\,dx + o(1)$$

and

$$\int_{\mathbb{R}^{\mathbb{N}}} K(x)|u_n|^{2^*}\,dx = \int_{\mathbb{R}^{\mathbb{N}}} K(x)|v_n|^{2^*}\,dx + \int_{\mathbb{R}^{\mathbb{N}}} K(x)|v|^{2^*}\,dx + o(1).$$

Then

$$\begin{aligned}\langle I_\lambda'(u_n), u_n\rangle &= \int_{\mathbb{R}^{\mathbb{N}}} \big[|\nabla u_n|^2 + \big(V(x)-\lambda\big)u_n^2\big]\,dx - \int_{\mathbb{R}^{\mathbb{N}}} a(x)|u_n|^p\,dx - \int_{\mathbb{R}^{\mathbb{N}}} K(x)|u_n|^{2^*}\,dx \\ &= \int_{\mathbb{R}^{\mathbb{N}}} \big[|\nabla v_n|^2 + (1-\lambda)v_n^2\big]\,dx + \int_{\mathbb{R}^{\mathbb{N}}} \big[|\nabla v|^2 + \big(V(x)-\lambda\big)v^2\big]\,dx \\ &\quad - \int_{\mathbb{R}^{\mathbb{N}}} |v_n|^p\,dx - \int_{\mathbb{R}^{\mathbb{N}}} a(x)|v|^p\,dx - \int_{\mathbb{R}^{\mathbb{N}}} K(x)|v_n|^{2^*}\,dx - \int_{\mathbb{R}^{\mathbb{N}}} K(x)|v|^{2^*}\,dx + o(1).\end{aligned}$$

By (B.10) and $\langle I_\lambda'(u_n), u_n\rangle \to 0$, we may therefore assume that

$$\int_{\mathbb{R}^{\mathbb{N}}} \big[|\nabla v_n|^2 + (1-\lambda)v_n^2\big]\,dx \to d$$

and

$$\int_{\mathbb{R}^{\mathbb{N}}} a(x)|v_n|^p\,dx + \int_{\mathbb{R}^{\mathbb{N}}} K(x)|v_n|^{2^*}\,dx \to d,$$

where $d > 0$.

Case (i): $b \neq 0$. Similarly to the proof of Lemma (3.2), we obtain

$$I_\lambda(u_n) \geqslant \mathfrak{S},$$

which contradict with $I_\lambda(u_n) < \varepsilon_0 < \mathfrak{S}$.

Case (ii): If $b = 0$, i.e., $v_n \to 0$ in $H^1(\mathbb{R}^N)$, then $I_\lambda(v) = \lim_{n\to\infty} I_\lambda(u_n) \leqslant \varepsilon'' < \varepsilon_1$. From (B.10) and Lemma 4.2, $v = 0$. So $0 = I_\lambda(v) = \lim_{n\to\infty} I_\lambda(u_n) \geqslant \varepsilon' > 0$. This is a contradiction also.

We complete the proof.

References

[1] S. Alama, Y.Y. Li, Existence of solutions for semilinear elliptic equations with indefinite linear part, J. Differential Equations 96 (1992) 89–115.

[2] A. Bahri, Y.Y. Li, On a minmax procedure for the existence of a positive solution for some certain scalar field equations in $\mathbb{R}^N$, Rev. Mat. Iberoamericana 6 (1990) 1–15.

[3] T. Bartsch, Y.H. Ding, On a nonlinear Schrödinger equation with periodic potential, Math. Ann. 313 (1999) 15–37.

[4] B. Buffoni, L. Jeanjean, C.A. Stuart, Existence of nontrivial solutions to a strongly indefinite semilinear equation, Proc. Amer. Math. Soc. 119 (1993) 179–186.

[5] J. Chabrowski, A. Szulkin, On a semilinear Schrödinger equation with critical Sobolev exponent, Proc. Amer. Math. Soc. 130 (2002) 85–93.

[6] D.G. Costa, H. Tehrani, Existence and multiplicity results for a class of Schrödinger equations with indefinite nonlinearities, Adv. Differential Equations 8 (2003) 1319–1340.

[7] Y. Egorov, V. Kondratiev, On Spectral Theory of Elliptic Operators, Birkhäuser, Basel, 1996.

[8] F. Gazzola, M. Lazzarino, Existence results for general critical growth semilinear elliptic equations, Commun. Appl. Anal. 4 (2000) 39–50.

[9] B. Gidas, W.M. Ni, L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^N$, in: L. Nachbin (Ed.), Math. Anal. Appl. Part A, Adv. Math. Suppl. Stud. Ser. A, vol. 7, Academic Press, New York, 1981, pp. 369–402.

[10] M.K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$, Arch. Ration. Mech. Anal. 105 (1989) 243–266.

[11] Y.Q. Li, Z.Q. Wang, J. Zeng, Ground states of nonlinear Schrödinger equations with potentials, Ann. Inst. H. Poincaré Anal. Non Linéaire 23 (2006) 829–837.

[12] P. Magrone, D. Mugnai, R. Servadei, Multiplicity of solutions for semilinear variational inequalities via linking and (∇)-theorems, J. Differential Equations 228 (2006) 191–225.

[13] A. Marino, C. Saccon, Some variational theorems of mixed type and elliptic problems with jumping nonlinearities, Ann. Sc. Norm. Super. Pisa Cl. Sc. (4) XXV (3–4) (1997) 661–665.

[14] D. Mugnai, Multiplicity of critical points in presence of a linking: Application to a superlinear boundary value problem, Nonlinear Differential Equations Appl. 11 (2004) 379–391.

[15] W.A. Strauss, Existence of solitary waves in higher dimensions, Commun. Math. Phys. 65 (1977) 149–162.

[16] F. Wang, Multiple solutions for some nonlinear Schrödinger equations with indefinite linear part, J. Math. Anal. Appl. 331 (2007) 1001–1022.

[17] F. Wang, Y.S. Huang, On a semi-linear Schrödinger equation in $\mathbb{R}^N$, Nonlinear Anal. 62 (2005) 833–848.

[18] M. Willem, Minimax Theorems, Birkhäuser, Boston, 1996.